ON GEOMETRY OF THE RÖSSLER SYSTEM OF EQUATIONS

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Abstract

Theory of Riemann extensions of spaces of constant affine connection is proposed to study the Rössler system of equations

$$\frac{dx}{ds} = -(y+z), \quad \frac{dy}{ds} = x + \alpha y, \quad \frac{dz}{ds} = \beta + xz - \nu z.$$

After its presentation in a homogeneous form

$$\frac{d}{ds}\xi(s) + 1/5\,\xi^2 - 1/5\,\xi\,\rho\,\nu + 1/5\,\xi\,\rho\,\alpha + \eta\,\rho + \theta\,\rho = 0$$

$$\frac{d}{ds}\eta(s) + 1/5\,\xi\,\eta - \xi\,\rho - 1/5\,\eta\,\rho\,\nu - 4/5\,\eta\,\rho\,\alpha = 0$$

$$\frac{d}{ds}\theta(s) - 4/5\,\xi\,\theta + 4/5\,\theta\,\rho\,\nu + 1/5\,\theta\,\rho\,\alpha - \beta\,\rho^2 = 0$$

$$\frac{d}{ds}\rho(s) + 1/5\,\xi\,\rho - 1/5\,\rho^2\nu + 1/5\,\rho^2\alpha = 0,$$

where

$$x = \frac{\xi}{\rho}, \quad y = \frac{\eta}{\rho}, \quad z = \frac{\theta}{\rho},$$

it can be considered as geodesic equations

$$\frac{d^2X^i}{ds^2} + \Pi^i_{jk}\frac{dX^j}{ds}\frac{dX^k}{ds} = 0.$$

of four dimensional space M^4 of constant affine connection with the components $\Pi^i_{jk} = \Pi^i_{jk}(\alpha, \beta, \nu)$ depending on parameters.

1 From the first order system of equations to the second order systems of ODE

The systems of the first order differential equations

$$\frac{dx^i}{ds} = c^i + a^i_j x^j + b^i_{jk} x^j x^k \tag{1}$$

depending on the parameters a, b, c are not suitable object of consideration from usually point of the Riemann geometry.

The systems of the second order differential equations in form

$$\frac{d^2x^i}{ds^2}+\Pi^i_{kj}(x)\frac{dx^k}{ds}\frac{dx^j}{ds}=0 \tag{2}$$

are best suited to do that.

They can be considered as geodesic of affinely connected space M^k in local coordinates x^k. The values $\Pi^i_{jk}=\Pi^i_{kj}$ are the coefficients of affine connections on M^k.

With the help of such coefficients can be constructed curvature tensor and others geometrical objects defined on variety M^k.

2 From affinely connected space to the Riemann space

We shall construct the Riemann space starting from a given affinely connected space defined by the systems of the second order ODE's.

With this aim we use the notion of the Riemann extension of nonriemannian space which was used earlier in the articles of author.

Remind the basic properties of this construction.

With help of the coefficients of affine connection of a given n-dimensional space can be introduced 2n-dimensional Riemann space D^{2n} in local coordinates (x^i,Ψ_i) having the metric of form

$$^{2n}ds^2=-2\Pi^k_{ij}(x^l)\Psi_k dx^i dx^j+2d\Psi_k dx^k \tag{3}$$

where Ψ_k are an additional coordinates.

The important property of such type metric is that the geodesic equations of metric (3) decomposes into two parts

$$\ddot{x}^k+\Pi^k_{ij}\dot{x}^i\dot{x}^j=0, \tag{4}$$

and

$$\frac{\delta^2\Psi_k}{ds^2}+R^l_{kji}\dot{x}^j\dot{x}^i\Psi_l=0, \tag{5}$$

where

$$\frac{\delta\Psi_k}{ds}=\frac{d\Psi_k}{ds}-\Pi^l_{jk}\Psi_l\frac{dx^j}{ds}$$

and R^l_{kji} are the curvature tensor of n-dimensional space with a given affine connection.

The first part (4) of the full system is the system of equations for geodesic of basic space with local coordinates x^i and it do not contains the supplementary coordinates Ψ_k.

The second part (5) of the system has the form of linear $N\times N$ matrix system of second order ODE's for supplementary coordinates Ψ_k

$$\frac{d^2\vec{\Psi}}{ds^2}+A(s)\frac{d\vec{\Psi}}{ds}+B(s)\vec{\Psi}=0. \tag{6}$$

Remark that the full system of geodesics has the first integral

$$-2\Pi^k_{ij}(x^l)\Psi_k\frac{dx^i}{ds}\frac{dx^j}{ds}+2\frac{d\Psi_k}{ds}\frac{dx^k}{ds}=\nu \tag{7}$$

which is equivalent to the relation

$$2\Psi_k\frac{dx^k}{ds}=\nu s+\mu \tag{8}$$

where μ,ν are parameters.

The geometry of extended space connects with geometry of basic space. For example the property of the space to be Ricci-flat $R_{ij}=0$ or symmetrical $R_{ijkl;m}=0$ keeps also for an extended space.

It is important to note that for extended space having the metric (3) all scalar curvature invariants are vanished.

As consequence the properties of linear system of equation (5-6) depending from the the invariants of $N \times N$ matrix-function

$$E = B - \frac{1}{2}\frac{dA}{ds} - \frac{1}{4}A^2$$

under change of the coordinates Ψ_k can be of used for that.

First applications the notion of extended spaces for the studying of nonlinear second order ODE's connected with nonlinear dynamical systems have been considered by author (V.Dryuma 2000-2008).

3 Eight-dimensional Riemann space for the Rössler system of equations

To investigation the properties of the Rössler system equations

$$\frac{dx}{ds} = -(y+z), \quad \frac{dy}{ds} = x + \alpha y, \quad \frac{dz}{ds} = \beta + xz - \nu z \tag{9}$$

we use its presentation in homogeneous form

$$\frac{d}{ds}\xi(s) + 1/5\,\xi^2 - 1/5\,\xi\,\rho\,\nu + 1/5\,\xi\,\rho\,\alpha + \eta\,\rho + \theta\,\rho = 0$$

$$\frac{d}{ds}\eta(s) + 1/5\,\xi\,\eta - \xi\,\rho - 1/5\,\eta\,\rho\,\nu - 4/5\,\eta\,\rho\,\alpha = 0$$

$$\frac{d}{ds}\theta(s) - 4/5\,\xi\,\theta + 4/5\,\theta\,\rho\,\nu + 1/5\,\theta\,\rho\,\alpha - \beta\,\rho^2 = 0$$

$$\frac{d}{ds}\rho(s) + 1/5\,\xi\,\rho - 1/5\,\rho^2\nu + 1/5\,\rho^2\alpha = 0,$$

where

$$x = \frac{\xi}{\rho}, \quad y = \frac{\eta}{\rho}, \quad z = \frac{\theta}{\rho}.$$

The relation between both systems is defined by the conditions

$$x(s) = \frac{\xi}{\rho}, \quad y(s) = \frac{\eta}{\rho}, \quad z(s) = \frac{\theta}{\rho}.$$

Remark that for a given system

$$\dot{\xi} = P, \quad \dot{\eta} = Q, \quad \dot{\theta} = R, \quad \dot{\rho} = T$$

the condition

$$P_\xi + Q_\eta + R_\theta + T_\rho = 0$$

is fulfield.

Such type of the system can be rewriten in the form

$$\frac{d^2X^i}{ds^2} + \Pi^i_{jk}\frac{dX^j}{ds}\frac{dX^k}{ds} = 0,$$

which allow us to consider it as geodesic equations of the space with constant affine connection.

In our case nonzero components of connection are

$$\Pi^1_{11} = \frac{1}{5}, \quad \Pi^1_{14} = \frac{\alpha-\nu}{10}, \quad \Pi^1_{24} = \frac{1}{2}, \quad \Pi^1_{34} = \frac{1}{2}, \quad \Pi^2_{12} = \frac{1}{10},$$

$$\Pi^2_{14} = -\frac{1}{2}, \quad \Pi^2_{24} = -\frac{4\alpha+\nu}{10}, \quad \Pi^3_{34} = \frac{4\nu+\alpha}{10},$$
$$\Pi^3_{12} = -\frac{4}{10}, \quad \Pi^3_{44} = -\beta, \quad \Pi^4_{14} = \frac{1}{10}, \quad \Pi^4_{44} = \frac{\alpha-\nu}{5}.$$

The metric of corresponding Riemann space is

$$\begin{gathered} {}^8ds^2 = -2\,\Pi^1_{11}\,Pdx^2 + 2\,\left(-2\,\Pi^2_{12}\,Q - 2\,\Pi^3_{12}\,U\right)dxdy + 2\,\left(-2\,\Pi^1_{14}\,P - 2\,\Pi^2_{14}\,Q - 2\,\Pi^4_{14}\,V\right)dxdu + \\ +2\,\left(-2\,\Pi^1_{24}\,P - 2\,\Pi^2_{24}\,Q\right)dydu + 2\,\left(-2\,\Pi^1_{34}\,P - 2\,\Pi^3_{34}\,U\right)dzdu + \\ + \left(-2\,\Pi^3_{44}\,U - 2\,\Pi^4_{44}\,V\right)du^2 + 2\,dxdP + 2\,dydQ + 2\,dzdU + 2\,dudV, \end{gathered} \tag{10}$$

where (P, Q, U, V) are an additional coordinates.

Geodesic of the metric (10) for coordinates (x, y, z, u) are

$$\frac{d^2}{ds^2}x(s) + 1/5\left(\frac{d}{ds}x(s)\right)^2 + 1/5\left(\frac{d}{ds}u(s)\right)\left(\frac{d}{ds}x(s)\right)\alpha - 1/5\left(\frac{d}{ds}u(s)\right)\left(\frac{d}{ds}x(s)\right)\nu + \left(\frac{d}{ds}u(s)\right)\frac{d}{ds}y(s) +$$
$$+\left(\frac{d}{ds}u(s)\right)\frac{d}{ds}z(s) = 0,$$
$$\frac{d^2}{ds^2}y(s) + 1/5\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}x(s) - \left(\frac{d}{ds}u(s)\right)\frac{d}{ds}x(s) - 4/5\left(\frac{d}{ds}u(s)\right)\left(\frac{d}{ds}y(s)\right)\alpha - 1/5\left(\frac{d}{ds}u(s)\right)\left(\frac{d}{ds}y(s)\right)\nu = 0,$$
$$\frac{d^2}{ds^2}z(s) - 4/5\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}x(s) + 4/5\left(\frac{d}{ds}u(s)\right)\left(\frac{d}{ds}z(s)\right)\nu + 1/5\left(\frac{d}{ds}u(s)\right)\left(\frac{d}{ds}z(s)\right)\alpha - \beta\left(\frac{d}{ds}u(s)\right)^2 = 0,$$
$$\frac{d^2}{ds^2}u(s) + 1/5\left(\frac{d}{ds}u(s)\right)\frac{d}{ds}x(s) + 1/5\left(\frac{d}{ds}u(s)\right)^2\alpha - 1/5\left(\frac{d}{ds}u(s)\right)^2\nu = 0$$

and they have a form of homogeneous Rössler system in the variables

$$\xi = \frac{dx}{ds}, \quad \eta = \frac{dy}{ds}, \theta = \frac{dz}{ds}, \quad \rho = \frac{du}{ds}.$$

The system of second order differential equations for additional coordinates can be reduced to the linear system of the first order equations with variable coefficients

$$\frac{d}{dt}P(t) + A_1P(t) + B_1Q(t) + C_1U(t) + E_1V(t) = 0,$$
$$\frac{d}{dt}Q(t) + A_2P(t) + B_2Q(t) + C_2U(t) + E_2V(t) = 0,$$
$$\frac{d}{dt}U(t) + A_3P(t) + B_3Q(t) + C_3U(t) + E_3V(t) = 0,$$
$$\frac{d}{dt}V(t) + A_4P(t) + B_4Q(t) + C_4U(t) + E_4V(t) = 0,$$

where $A_i,\ B_i,\ C_i,\ E_i$ are the functions of the variables $x,\ y,\ z,\ u)$.

Properties of such type of the systems can be investigated with help of the Wilczynski invariants.

4 Laplace operator

In theory of Riemann spaces the equation

$$L\psi = g^{ij}(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k})\psi(x) = 0 \qquad (11)$$

can be used to the study of the properties of spaces.

For the eight-dimensional space with the metric (10) corresponded the Rössler system we get the equation on the function $\psi(x,y,z,u,P,Q,U,V) = \theta(P,Q,U,V)$

$$4/5\,\frac{\partial}{\partial P}\theta(P,Q,U,V) + 2/5\,P\frac{\partial^2}{\partial P^2}\theta(P,Q,U,V) - 8/5\left(\frac{\partial^2}{\partial P\partial Q}\theta(P,Q,U,V)\right)U+$$
$$+2/5\left(\frac{\partial^2}{\partial P\partial Q}\theta(P,Q,U,V)\right)Q - 2\left(\frac{\partial^2}{\partial P\partial V}\theta(P,Q,U,V)\right)Q + 2/5\left(\frac{\partial^2}{\partial P\partial V}\theta(P,Q,U,V)\right)V+$$
$$+2\left(\frac{\partial^2}{\partial Q\partial V}\theta(P,Q,U,V)\right)P + 2\left(\frac{\partial^2}{\partial U\partial V}\theta(P,Q,U,V)\right)P + 8/5\left(\frac{\partial}{\partial V}\theta(P,Q,U,V)\right)\alpha+$$
$$+2/5\left(\frac{\partial}{\partial V}\theta(P,Q,U,V)\right)\nu - 2/5\left(\frac{\partial^2}{\partial V^2}\theta(P,Q,U,V)\right)V\nu + 2/5\left(\frac{\partial^2}{\partial V^2}\theta(P,Q,U,V)\right)V\alpha+$$
$$+2/5\left(\frac{\partial^2}{\partial U\partial V}\theta(P,Q,U,V)\right)U\alpha + 2/5\left(\frac{\partial^2}{\partial Q\partial V}\theta(P,Q,U,V)\right)Q\nu + 8/5\left(\frac{\partial^2}{\partial Q\partial V}\theta(P,Q,U,V)\right)Q\alpha-$$
$$-2/5\left(\frac{\partial^2}{\partial P\partial V}\theta(P,Q,U,V)\right)P\nu - 2\left(\frac{\partial^2}{\partial V^2}\theta(P,Q,U,V)\right)\beta\,U + 8/5\left(\frac{\partial^2}{\partial U\partial V}\theta(P,Q,U,V)\right)U\nu+$$
$$+2/5\left(\frac{\partial^2}{\partial P\partial V}\theta(P,Q,U,V)\right)P\alpha = 0.$$

This equation has varies type of particular solutions.

A simplest one is

$$\psi(x,y,z,u,P,Q,U,V) = e^{(\nu-\alpha)P - 5\,Q + 5\,U + V}$$

at the conditions on parameters of the Rössler system

$$\beta = \frac{25}{2}\,\nu, \quad \alpha = -3/2\,\nu.$$

As examples obtained by direct substitutions we get the quadratic solution

$$\theta(P,Q,U,V) =$$
$$= 1/18\,\frac{9\,Q^2 + 30\,QP\alpha - 18\,QV + 25\,P^2\alpha^2 - 30\,P\alpha\,V + 9\,V^2 + 15\,P^2 + 60\,U\alpha\,P + 45\,\beta\,UP}{\_c_2}$$

with conditions

$$\nu = 8/3\,\alpha, \quad \beta = arbitrary.$$

Cubic solution

$$\theta(P,Q,U,V) = -\frac{1}{240}\,\frac{\left(-96\,\alpha\,l3 - 2880\,\alpha^2 l1\right)V^3}{\beta^2} - \frac{1}{240}\,\frac{\left(-48\,\beta\,l3 - 1440\,\beta\,l1\,\alpha\right)UV^2}{\beta^2}+$$
$$+\left(-\frac{1}{240}\,\frac{\left(60\,\beta\,l3 + 1800\,\beta\,l1\,\alpha\right)P^2}{\beta^2} + l1\,U^2\right)V + k1\,U^3 + (l3\,P + l2\,Q)\,U^2-$$

$$-\frac{1}{240}\,\frac{\left(2250\,\beta\,l1\,\alpha+75\,\beta\,l3\right)P^2Q}{\beta^2}+\left(m2\,Q^2-\frac{1}{240}\,\frac{\left(400\,\beta^2 l1+3000\,\beta\,l1\,\alpha+100\,\beta\,l3\right)P^2}{\beta^2}\right)U+nQ^3$$

at the condition

$$\nu=6\alpha$$

and arbitrary coefficients l,m,k,n.

A polynomial solution of degree four

$$\theta(P,Q,U,V)=rQ^4+k1\,U^4-2/5\,m3\,U^3V+l2\,UQ^3+m2\,U^2Q^2+n2\,U^3Q+m3\,U^2P^2$$

at the condition

$$\nu=-\frac{7}{13}\,\alpha.$$

Remark that the properties of of such type of solutions depend on parameters and may be highly diversified.

More complicated solutions of the Laplace equation can be obtained by application of the method of (u,v)- transformation developed in the works of author.

5 Eikonal equation

Solutions of eikonal equation

$$g^{ij}\frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j}=0 \tag{12}$$

also gives useful information about the properties of Riemann space.

In particular the condition

$$F(x^1,x^2,...,x^i)=0$$

where function $F(x^i)$ satisfies the equation (12), determines $(N-1)$-dimensional hypersurface with normals forming an isotropic vector field.

For the space with the metric (10) the eikonal equation on the function $\psi(x,y,z,u,P,Q,U,V)=\eta(P,Q,U,V)$ takes the form

$$2/5\,P\left(\frac{\partial}{\partial P}\eta(P,Q,U,V)\right)^2-8/5\left(\frac{\partial}{\partial P}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial Q}\eta(P,Q,U,V)\right)U+$$

$$+2/5\left(\frac{\partial}{\partial P}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial Q}\eta(P,Q,U,V)\right)Q-2\left(\frac{\partial}{\partial P}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)Q+$$

$$+2/5\left(\frac{\partial}{\partial P}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)P\alpha-2/5\left(\frac{\partial}{\partial P}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)P\nu+$$

$$+2/5\left(\frac{\partial}{\partial P}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)V+2\left(\frac{\partial}{\partial Q}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)P-$$

$$-8/5\left(\frac{\partial}{\partial Q}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)Q\alpha-2/5\left(\frac{\partial}{\partial Q}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)Q\nu+$$

$$+2\left(\frac{\partial}{\partial U}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)P+8/5\left(\frac{\partial}{\partial U}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)U\nu+$$

$$+2/5\left(\frac{\partial}{\partial U}\eta(P,Q,U,V)\right)\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)U\alpha+2/5\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)^2V\alpha-$$

$$-2/5\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)^2V\nu-2\left(\frac{\partial}{\partial V}\eta(P,Q,U,V)\right)^2\beta\,U=0. \tag{13}$$

A simplest solution of this equation is

$$\eta(P,Q,U,V) = -\frac{Q}{\alpha} + \frac{V}{\nu-\alpha} + \frac{U}{\alpha} + P$$

with condition on the coefficients of the Rössler system

$$-5\,\beta\,\alpha - 11\,\alpha\,\nu + 8\,\nu^2 + 3\,\alpha^2 = 0. \tag{14}$$

From here we find

$$\nu = \frac{11}{16}\,\alpha + 1/16\,\sqrt{25\,\alpha^2 + 160\,\beta\,\alpha}$$

To provide a more complicated solutions of the equation (12) we use the method of (u,v)-transformation. For the sake of convenience we rewrite the equation (12) in the form

$$2/5\,x\left(\frac{\partial}{\partial x}\eta(x,y,z,p)\right)^2 - 8/5\left(\frac{\partial}{\partial x}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial y}\eta(x,y,z,p)\right)z+$$
$$+2/5\left(\frac{\partial}{\partial x}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial y}\eta(x,y,z,p)\right)y - 2\left(\frac{\partial}{\partial x}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)y+$$
$$+2/5\left(\frac{\partial}{\partial x}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)x\alpha - 2/5\left(\frac{\partial}{\partial x}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)x\nu+$$
$$+2/5\left(\frac{\partial}{\partial x}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)p + 2\left(\frac{\partial}{\partial y}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)x-$$
$$-8/5\left(\frac{\partial}{\partial y}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)y\alpha - 2/5\left(\frac{\partial}{\partial y}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)y\nu+$$
$$+2\left(\frac{\partial}{\partial z}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)x + 8/5\left(\frac{\partial}{\partial z}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)z\nu+$$
$$+2/5\left(\frac{\partial}{\partial z}\eta(x,y,z,p)\right)\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)z\alpha + 2/5\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)^2 p\alpha - 2/5\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)^2 p\nu-$$
$$-2\left(\frac{\partial}{\partial p}\eta(x,y,z,p)\right)^2\beta\,z = 0 \tag{15}$$

Now after change of the function and derivatives in accordance with the rules

$$\eta(x,y,z,p) \to u(x,t,z,p), \quad y \to v(x,t,z,p,$$

$$\frac{\partial\eta(x,y,z,p)}{\partial x} \to \frac{\partial u(x,t,z,p)}{\partial x} - \frac{\frac{\partial u(x,t,z,p)}{\partial t}}{\frac{\partial v(x,t,z,p))}{\partial t}}\frac{\partial v(x,t,z,p)}{\partial x},$$

$$\frac{\partial\eta(x,y,z,p)}{\partial z} \to \frac{\partial u(x,t,z,p)}{\partial z} - \frac{\frac{\partial u(x,t,z,p)}{\partial t}}{\frac{\partial v(x,t,z,p))}{\partial t}}\frac{\partial v(x,t,z,p)}{\partial z},$$

$$\frac{\partial\eta(x,y,z,p)}{\partial p} \to \frac{\partial u(x,t,z,p)}{\partial p} - \frac{\frac{\partial u(x,t,z,p)}{\partial t}}{\frac{\partial v(x,t,z,p))}{\partial t}}\frac{\partial v(x,t,z,p)}{\partial p},$$

$$\frac{\partial\eta(x,y,z,p)}{\partial y} \to \frac{\frac{\partial u(x,t,z,p)}{\partial t}}{\frac{\partial v(x,t,z,p))}{\partial t}},$$

where

$$u(x,t,z,p) = t\frac{\partial}{\partial t}\omega(x,t,z,p) - \omega(x,t,z,p), \quad v(x,t,z,p) = \frac{\partial}{\partial t}\omega(x,t,z,p)$$

we find the equation on the function $\omega(x,t,z,p)$

$$\left(\frac{\partial}{\partial x}\omega(x,t,z,p)\right)^2 x - 5\,tx\frac{\partial}{\partial p}\omega(x,t,z,p) - p\nu\left(\frac{\partial}{\partial p}\omega(x,t,z,p)\right)^2 - 5\,\beta\,z\left(\frac{\partial}{\partial p}\omega(x,t,z,p)\right)^2 +$$

$$+p\left(\frac{\partial}{\partial x}\omega(x,t,z,p)\right)\frac{\partial}{\partial p}\omega(x,t,z,p) + 4\,z\nu\left(\frac{\partial}{\partial z}\omega(x,t,z,p)\right)\frac{\partial}{\partial p}\omega(x,t,z,p) + 4\left(\frac{\partial}{\partial x}\omega(x,t,z,p)\right)zt -$$

$$-\left(\frac{\partial}{\partial x}\omega(x,t,z,p)\right)\left(\frac{\partial}{\partial t}\omega(x,t,z,p)\right)t - 5\left(\frac{\partial}{\partial t}\omega(x,t,z,p)\right)\left(\frac{\partial}{\partial x}\omega(x,t,z,p)\right)\frac{\partial}{\partial p}\omega(x,t,z,p) +$$

$$+p\alpha\left(\frac{\partial}{\partial p}\omega(x,t,z,p)\right)^2 + 5\,x\left(\frac{\partial}{\partial z}\omega(x,t,z,p)\right)\frac{\partial}{\partial p}\omega(x,t,z,p) + z\alpha\left(\frac{\partial}{\partial z}\omega(x,t,z,p)\right)\frac{\partial}{\partial p}\omega(x,t,z,p) +$$

$$+4\,t\left(\frac{\partial}{\partial t}\omega(x,t,z,p)\right)\alpha\frac{\partial}{\partial p}\omega(x,t,z,p) + t\left(\frac{\partial}{\partial t}\omega(x,t,z,p)\right)\nu\frac{\partial}{\partial p}\omega(x,t,z,p) -$$

$$-x\nu\left(\frac{\partial}{\partial x}\omega(x,t,z,p)\right)\frac{\partial}{\partial p}\omega(x,t,z,p) + x\alpha\left(\frac{\partial}{\partial x}\omega(x,t,z,p)\right)\frac{\partial}{\partial p}\omega(x,t,z,p) = 0. \qquad (16)$$

In spite of the fact that this equation looks not a simple than equation (15) its particular solutions can be find without trouble.

As example the substitution of the form

$$\omega(x,t,z,p) = B(x,t,z) + kpt,$$

into the equation (16) lead to expression on the function $B(x,t,z)$

$$B(x,t,z) = \alpha\,tx + tz + A(t),$$

with arbitrary function $A(t)$ and the conditions on coefficients of the Rössler system

$$\beta = 1/5\,\frac{\alpha\,(8+5\,k)}{k^2}, \quad \nu = \frac{\alpha\,(1+k)}{k}$$

depending from arbitrary parameter k.

Using the function $\omega(x,t,z,p)$ we can obtain the solution of the equation (13) by elimination of the parameter t from the relations

$$\eta(x,y,z,p) - t\frac{\partial}{\partial t}\omega(x,t,z,p) + \omega(x,t,z,p) = 0, \quad y - \frac{\partial}{\partial t}\omega(x,t,z,p) = 0.$$

As example at the choice ($A(t) = 1 + t^2$ we get

$$\eta(x,y,z,p) = 1/4\,k^2p^2 + (-1/2\,yk + 1/2\,zk + 1/2\,\alpha\,xk)\,p + 1/4\,z^2 + (1/2\,\alpha\,x - 1/2\,y)\,z - 1 + 1/4\,y^2 - 1/2\,y\alpha\,x + 1/4\,\alpha^2x^2.$$

At the condition $A(t) = \ln(t)$ we get

$$\eta(x,y,z,p) = 1 - \ln(-(-y + \alpha\,x + z + kp)^{-1}).$$

Remark that parameters α, β, ν in these cases the relation (14) is satisfied.

To cite another example.

Substitution the expression

$$\omega(x,t,z,p) = A(t)x + tz + C(t)p$$

into the equation (16) lead to the system of equations on the functions $A(t),\ C(t)$

$$(t\nu\,C(t) + 4\,t\alpha\,C(t) - 5\,A(t)C(t) - A(t)t)\,\frac{d}{dt}A(t) + (A(t))^2 - \nu\,A(t)C(t) + \alpha\,A(t)C(t) = 0,$$

$$(t\nu\, C(t) + 4\, t\alpha\, C(t) - 5\, A(t)C(t) - A(t)t)\, \frac{d}{dt}C(t) + A(t)C(t) - \nu\, (C(t))^2 + \alpha\, (C(t))^2 = 0,$$

$$-5\,\beta\, z\, (C(t))^2 + (5\, z\alpha\, t - 5\, A(t)z + 5\, z\nu\, t)\, C(t) + 3\, A(t)zt = 0.$$

From this system of equations we find the condition on parameters

$$\nu = 3/5\,\beta - \alpha \tag{17}$$

and expression on the function $A(t)$

$$A(t) = -\beta\, C(t).$$

Function $C(t)$ in this case satisfies the equation

$$(25\,\beta\, C(t) + 8\,\beta\, t + 15\, t\alpha)\, \frac{d}{dt}C(t) - 8\,\beta\, C(t) + 10\,\alpha\, C(t) = 0.$$

Its solution is defined by the relation

$$-t\alpha + (C(t))^{-1/2\, \frac{8\,\beta + 15\,\alpha}{-4\,\beta + 5\,\alpha}}\, \_C1\,\alpha - \beta\, C(t) = 0.$$

Using the expression on the function $C(t)$ we can find the function $\omega(x,t,z,p)$ and after elimination of the parameter t from corresponding relations it is possible to get the solution of the eikonal equation at the condition (17) on parameters of the Rössler system of equations.

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